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On Rellich's Inequalities in Euclidean Spaces

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Abstract—On domains of Euclidean spaces we consider inequalities for test functions and their Laplacians. We describe a family of domains having vanishing Rellich constants. For the Euclidean space of dimension 4 we present a new version of the Rellich inequality. In addition, we prove new one-dimensional Rellich-type integral inequalities for linear combinations of test functions and their derivatives of orders one and two.

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Introduction. In books [1] and [2] one can find basic results on the Rellich inequalities in Euclidean spaces. We will need the following F. Rellich's inequality [1] with sharp constants:

$$\int_{\mathbb{R}^n} |\Delta f|^2\,dx \geq \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n}\frac{|f|^2}{|x|^4}dx \quad \forall f\in C_0^\infty(\mathbb{R}^n\setminus\{0\}), \tag{1}$$

where $n\in\mathbb{N}$, $n\neq 2$. There are several generalizations of this inequality (see [3–8] and the bibliography therein). The known results are connected with some direct analogs of (1), when one compares integrals for functions of the type $|x|^{\alpha+4}|\Delta f|^2$ and $|x|^\alpha|f|^2$, as well as there are results when the Laplace operator is replaced by polyharmonic operators. Consider now an analog of inequality (1) for smooth test functions $f:\Omega\to\mathbb{R}$, defined in the domain $\Omega\subset\mathbb{R}^n$, i.e., in an open connected subset of the Euclidean space. Let $\Omega\neq\mathbb{R}^n$ and let $\partial\Omega$ be the boundary of the domain Ω. Then for every $x\in\Omega$ it is well-defined the distance $\operatorname{dist}(x,\partial\Omega)=\inf\limits_{y\in\partial\Omega}|x-y|$. Suppose that $\mu\in\mathbb{R}$. We will consider the inequality

$$\int_\Omega \frac{|\Delta f|^2\,dx}{\operatorname{dist}^{2\mu}(x,\partial\Omega)} \geq C_{2\mu}(\Omega)\int_\Omega\frac{|f|^2\,dx}{\operatorname{dist}^{4+2\mu}(x,\partial\Omega)} \quad \forall f\in C_0^\infty(\Omega), \tag{2}$$

where the constant $C_{2\mu}(\Omega)$ is defined as the maximum possible constant at this place, i.e.,

$$C_{2\mu}(\Omega)=\inf_{f\in C_0^\infty(\Omega),\ f\not\equiv 0}\frac{\int\limits_\Omega |\Delta f|^2\operatorname{dist}^{-2\mu}(x,\partial\Omega)\,dx}{\int\limits_\Omega |f|^2\operatorname{dist}^{-4-2\mu}(x,\partial\Omega)\,dx}.$$

Since $\operatorname{dist}(x,\partial(\mathbb{R}^n\setminus\{0\}))=|x|$ and the constant in Rellich's inequality (1) is sharp, one has that $C_0(\mathbb{R}^n\setminus\{0\})=n^2(n-4)^2/16$ for $n\in\mathbb{N}\setminus\{2\}$. The sharp constant $C_{2\mu}((\mathbb{R}^n\setminus\{0\})$ for all values of $\mu\in\mathbb{R}$ is defined by the following formula of Caldiroli and Musina [3]:

$$C_{2\mu}(\mathbb{R}^n\setminus\{0\})=\min_{k\in\mathbb{N}\cup\{0\}}\left|\frac{(n-2)^2}{4}-(\mu+1)^2+k(n-2+k)\right|^2. \tag{3}$$

Main results. From the definition of the constant $C_{2\mu}(\Omega)$ it follows that $C_{2\mu}(\Omega)\in[0,\infty)$ for every domain Ω and every value of $\mu\in\mathbb{R}$.

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